SIMPLICIAL COMPLEXES: EMERGENT HYPERBOLIC NETWORK GEOMETRY AND FRUSTRATED SYNCHRONIZATION

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Network Topology and Network Geometry

are expected to have impact in a variety of applications,

ranging from

brain research to routing protocols in the Internet
Community Structure and Network Topology

most complex networks have a mesoscale structure which reveal densely connected communities

From S. Fortunato RMP
The role of dimensionality in neuronal dynamics

Uloa Severino et al. Scientific Reports (2016)
Generalized network structures

Going beyond the framework of simple networks is of fundamental importance for understanding the relation between structure and dynamics in complex systems.
Simplicial complexes are characterizing the interaction between two or more nodes and are formed by nodes, links, triangles, tetrahedra etc.
Brain data as simplicial complexes

Giusti et al (2016)
Protein interaction networks as simplicial complexes

Protein interaction networks

- **Nodes:** proteins
- **Simplices:** protein complexes

Wan et al. Nature 2015
Collaboration networks as simplicial complexes

Actor collaboration networks

• **Nodes:** Actors
• **Simplicies:** Co-actors of a movie

Scientific collaboration networks

• **Nodes:** Scientists
• **Simplicies:** Co-authors
The hidden metric of complex networks

It is believed that most complex networks have an hidden metric such that the nodes close in the hidden metric are more likely to be linked to each other.
Emergent geometry

In the framework of emergent geometry, networks with hidden geometry are generated by equilibrium or non-equilibrium dynamics that makes no use of the hidden geometry.
Growing networks describe the emergence of scale-free networks

Would growing simplicial complexes describe the emergence of hyperbolic complex network geometry?
The generalized degree $k_{d,\delta}(\mu)$ of a $\delta$-face $\mu$ in a $d$-dimensional simplicial complex is given by the number of $d$-dimensional simplices incident to the $\delta$-face $\mu$. 

Number of triangles incident to the node $\mu$

$k_{2,0}(\mu)$

Number of triangles incident to the link $\mu$

$k_{2,1}(\mu)$
Generalized degree

The generalized degree $k_{d,\delta}(\mu)$ of a $\delta$-face $\mu$ in a $d$-dimensional simplicial complex is given by the number of $d$-dimensional simplices incident to the $\delta$-face $\mu$. 

<table>
<thead>
<tr>
<th>$i$</th>
<th>$k_{2,0}(i)$</th>
<th>$(i,j)$</th>
<th>$k_{2,1}(i,j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>(1,2)</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>(1,3)</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>(1,4)</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>(1,5)</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>(2,3)</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>(3,4)</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3,5)</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3,6)</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(5,6)</td>
<td>1</td>
</tr>
</tbody>
</table>
Incidence number

To each (d-1)-face $\mu$ we associate the incidence number

$$n_\mu = k_{d,d-1}(\mu) - 1$$
Manifolds

If $n_\mu$ takes only values $n_\mu = 0, 1$ each $(d-1)$-face is incident at most to two $d$-dimensional simplices.

*In this case the simplicial complex is a discrete manifold.*

![Diagram showing a complex that is not a manifold on the left and a manifold on the right.](image)
Network Geometry with Flavor

Starting from a single $d$-dimensional simplex

1) **GROWTH**: At every timestep we add a new $d$ simplex (formed by one new node and an existing $(d-1)$-face).

2) **ATTACHMENT**: The probability that a new node will be connected to a face $\mu$ depends on the flavor $s=-1,0,1$ and is given by

$$\prod_{\mu}^{[s]} = \frac{1 + s n_{\mu}}{\sum_{\mu'} (1 + s n_{\mu'})}$$

Bianconi & Rahmede (2016)
Attachment probability

\[
\Pi^{[s]}_\mu = \frac{(1 + s n_\mu)}{\sum_{\mu' \in Q_{d,d-1}} (1 + s n_{\mu'})} = \begin{cases} 
\frac{(1 - n_\mu)}{Z^{[-1]}}, & s = -1 \\
\frac{1}{Z^{[0]}}, & s = 0 \\
\frac{k_\mu}{Z^{[1]}}, & s = 1
\end{cases}
\]

\[s = -1 \quad \text{Manifold} \quad n_\mu = 0, 1\]
\[s = 0 \quad \text{Uniform attachment} \quad n_\mu = 0, 1, 2, 3, 4, \ldots\]
\[s = 1 \quad \text{Preferential attachment} \quad n_\mu = 0, 1, 2, 3, 4, \ldots\]
Dimension $d=1$

- Manifold
- Uniform attachment
- Preferential attachment

Chain  Exponential  Scale-free BA model
Dimension $d=2$

Manifold  Uniform attachment  Preferential attachment

Exponential  Scale-free  Scale-free
Dimension $d=3$

- Manifold
- Uniform attachment
- Preferential attachment

Scale-free  Scale-free  Scale-free
Effective preferential attachment in $d=3$

Node $i$ has generalized degree 3
Node $i$ is incident to 5 unsaturated faces

Node $i$ has generalized degree 4
Node $i$ is incident to 6 unsaturated faces
Degree distribution

For $d+s=1$

$$P_d(k) = \left( \frac{d}{d+1} \right)^{k-d} \frac{1}{d+1}$$

For $d+s>1$

$$P_d(k) = \frac{d+s}{2d+s} \frac{\Gamma(1+(2s+s)(d+s-I))}{\Gamma(d/(d+s-I))} \frac{\Gamma(k-d+d/(d+s-I))}{\Gamma(k-d+(2d+s)(d+s-I))}$$

NGF are always scale-free for $d>1-s$

- For $s=1$ NGF are always scale free
- For $s=0$ and $d>1$ the NGF are scale-free
- For $s=-1$ and $d>2$ the NGF are scale-free
Degree distribution of NGF
Generalized degree distributions

<table>
<thead>
<tr>
<th>flavor</th>
<th>$s = -1$</th>
<th>$s = 0$</th>
<th>$s = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta = d - 1$</td>
<td>Bimodal</td>
<td>Exponential</td>
<td>Power-law</td>
</tr>
<tr>
<td>$\delta = d - 2$</td>
<td>Exponential</td>
<td>Power-law</td>
<td>Power-law</td>
</tr>
<tr>
<td>$\delta \leq d - 3$</td>
<td>Power-law</td>
<td>Power-law</td>
<td>Power-law</td>
</tr>
</tbody>
</table>

The power-law generalized degree distribution are scale-free for

$$d \geq d_{c}^{[\delta,s]} = 2(\delta + 1) + s$$
Emergent community structure

Modularity and Clustering coefficient of NGF

<table>
<thead>
<tr>
<th>M</th>
<th>s=−1</th>
<th>s=0</th>
<th>s=1</th>
<th>C</th>
<th>s=−1</th>
<th>s=0</th>
<th>s=1</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=2</td>
<td>0.97</td>
<td>0.94</td>
<td>0.90</td>
<td>d=2</td>
<td>0.65</td>
<td>0.74</td>
<td>0.79</td>
</tr>
<tr>
<td>d=3</td>
<td>0.91</td>
<td>0.85</td>
<td>0.80</td>
<td>d=3</td>
<td>0.77</td>
<td>0.81</td>
<td>0.84</td>
</tr>
</tbody>
</table>
Emergent Hyperbolic geometry

The emergent hidden geometry is the hyperbolic $H^d$ space
Here all the links have equal length

$d=2$
Emergent hyperbolic geometry

$d=3$
Apollonian networks are formed by linking the centers of an Apollonian sphere packing.
They are scale-free and are described by the Lorentz group.
Connection with the Apollonian network
Complex Network Manifolds And Frustrated Synchronization
Holography of Complex Network Manifolds

\[ d = 3 \quad D = 2 \]

\( d \)-dimensional Complex Network Manifolds can be interpreted as \( D \)-dimensional manifolds with \( D = d - 1 \)
Spectral dimensions of Complex Network Manifolds

\[ L_{ij} = \delta_{ij} - \frac{a_{ij}}{k_i} \]

\[ \rho_c(\lambda) \approx \lambda^{-d_s/2} \]

Complex Network Manifolds have finite spectral dimension with

\[ d_s \approx d \text{ for } d = 2, 3, 4 \]
Localization of the eigenvectors

$$\lambda_{\lambda} = \left( \sum_{i=1}^{N} (u_i \lambda v_i) \right)^{-1}$$

The participation ratio evaluates the effective number of nodes on which an eigenmode is localized.

A large number of eigenmodes are localized.
The Kuramoto model

We consider the Kuramoto model

\[ \frac{d\vartheta_i}{dt} = \omega_i + \sigma \sum_{j=1}^{N} \frac{a_{ij}}{k_i} \sin(\vartheta_j - \vartheta_i) \]

where \( \omega_i \) is the internal frequency of node \( i \) drawn randomly from a Gaussian distribution.

The global order parameter is

\[ R = \frac{1}{N} \sum_{j=1}^{N} e^{i\vartheta_j} \]
In an infinite fully connected network we have

\[ \text{Synchronized phase} \]
Frustrated synchronization

D=1

D=2

D=3

(a) (b) (c) (d) (e) (f)
Finite size effects

D=1

D=2

D=3

$N=100,200,400,800,1600,3200$

The finite size effects are less pronounced in larger dimensions
The fully synchronized phase is not thermodynamically achieved for networks with spectral dimension

\[ d_s \leq 4 \]

In Complex Network Manifolds with D=3 the fully synchronized state is marginally stable
Frustrated synchronization and community structure

For every community with $n_c$ nodes we can define the local order parameter

$$Z_{\text{mod}} = R_{\text{mod}} e^{i\psi_{\text{mod}}} = \frac{1}{n_c} \sum_{j \in C} e^{i\theta_j}$$
Communities and Frustrated Synchronization

(a) Im[Zmod] vs Re[Zmod]
(b) Im[Zmod] vs Re[Zmod]
(c) Im[Zmod] vs Re[Zmod]
(d) Rmod(t) vs t
(e) Rmod(t) vs t
(f) Rmod(t) vs t
(g) S(f) vs f
(h) S(f) vs f
(i) S(f) vs f
Localization of eigenvector on communities

\[ Y_Q = \left[ \sum_n \left( \sum_{i \in C_n} u_i^\lambda v_i^\lambda \right)^2 \right]^{-1} \]

measure in how many communities the eigenmode is localized

D=1

\[ P(Y_Q) \]

D=2

\[ P(Y_Q) \]

D=3

\[ P(Y_Q) \]
Correlations among communities and network coarse graining

\[ n_c = 70 \quad \text{and} \quad n_c = 30 \]

\[ N=1000, \ D=2, \ \sigma=5 \]
Conclusions

Complex Network Manifolds can help us understand the interplay between Network Topology, Network Geometry and Synchronization dynamics

Complex Network Manifolds and Frustrated Synchronization

• Complex Network Manifolds display Frustrated Synchronization with strong spatio-temporal fluctuations of the order parameter

• They combine small-world property and community structure like brain networks

• They show a strong dependence on the dimension with the fully synchronized state marginally stable in dimension $D=3$
Collaborators and References

Emergent network geometry

G. Bianconi C. Rahmede Network geometry with flavor PRE 93, 032315 (2016)
G. Bianconi and C. Rahmede Scientific Reports 7, 41974 (2017)
O. T. Courtney and G. Bianconi PRE 95, 062301 (2017)

Frustrated synchronization in Complex Network Manifolds

Ensembles of simplicial complexes
O. T. Courtney and G. Bianconi PRE 93, 062311 (2016)

CODES AVAILABLE AT GITHUB PAGE gimestone
Structure and Function
by
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• Pedagogical presentation

• Discussion of general concepts in terms of their impact on interdisciplinary applications